THE INFLUENCE OF VISCOSITY ANISOTROPY ON THE PLANE-PARALLEL FLOW OF AN IONIZED MEDIUM IN A COPLANAR MAGNETIC FIELD

E. G. Sakhnovskii

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 11–18, 1965

The author investigates the influence of anisotropy of the viscosity coefficients on the motion of an ionized medium between parallel plates with a magnetic field having components in the flow plane only. It is assumed that the compressibility of the medium may be neglected, as well as the temperature dependence of the transport coefficients, also that the degree of ionization may be regarded as constant. It is shown that taking Larmor rotation of the ions into account leads to a considerable complication of the flow pattern, as distinct from the case when the ions do not possess spiral paths, and the coplanar magnetic field exerts no influence on the motion of the medium. In particular, the viscous force introduces a transverse velocity component.

1. Statement of the problem. The motion of an incompressible medium which is a mixture of electron, ion, and neutral gases with a constant degree of ionization may be described by the following system of equations, derived in [1, 2]:

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u}\right] = -\nabla p - \operatorname{div}\pi + \mathbf{j}\times\mathbf{B}, \qquad \operatorname{div}\mathbf{u} = 0,$$

$$\mathbf{j} + \frac{\omega_e\tau_0}{B}\left(\mathbf{j}\times\mathbf{B} - \frac{Zs}{1+Zs}\nabla p\right) +$$

$$+ 2(1-s)^2\frac{\omega_i\tau_{ia}\omega_e\tau_0}{B^2}\Big[\mathbf{B}\times(\mathbf{j}\times\mathbf{B}) +$$

$$+ \frac{Zs}{1+Zs}\nabla p\times\mathbf{B} - \frac{1}{1-s}(s\operatorname{div}\pi - \operatorname{div}\pi_i)\times\mathbf{B}\Big] =$$

$$= \sigma_0(\mathbf{E} + \mathbf{u}\times\mathbf{B}),$$

$$\operatorname{rot}\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \qquad \operatorname{rot}\frac{\mathbf{B}}{\mu_0} = \mathbf{j}, \qquad \operatorname{div}\varepsilon_0\mathbf{E} = \rho_e, \qquad \operatorname{div}\mathbf{B} = 0$$

$$(\tau_0^{-1} = \tau_{ei}^{-1} + \tau_{ea}^{-1}). \qquad (1.1)$$

Here **u** is the velocity, p the pressure, ρ the density, s the degree of ionization of the medium; π and π_i are the viscous stress tensors of the mixture as a whole, and of its ionic component, expressions for which were obtained in [1]; B is the magnetic induction vector, E is the electric field strength vector, j is the current density vector, ρ_e is the volume charge, μ_0 and ε_0 are the magnetic and electric constants, ω_e and ω_i are the cyclotron frequencies of the electrons and ions, σ_0 = const is the conductivity of the medium without magnetic field, $\tau_{\alpha\beta}^{-1}$ is the effective collision frequency of particles of types α- and β- (α, β = e, i, a electron, ion, and neutral, respectively), and Z is the charge number.

We shall consider a channel formed by two infinite plates arranged in planes z = ± a, in which the motion of the ionized medium is induced either by a pressure differential in the direction of the x axis, or by the motion of the upper plate in the same direction (Couette flow). We shall assume that the electromagnetic field and the velocity depend on the transverse coordinate and time only d/dx = = d/dy = 0. Then from the hydrodynamic equation of continuity and the fact that the gas cannot penetrate the walls of the channel, we obtain $u_z \equiv 0$. From Maxwell's first equation and the boundary condition implying the absence of a normal component of the magnetic induction vector we have $B_z \equiv 0$. Finally, from Maxwell's second equation* we have $j_z \equiv 0$. We write the remaining components of equations (1.1) on the coordinate axes in dimensionless form, and to do this we introduce, from considerations of the physics of the problem, the quantities

$$z,\ \mathbf{u},\ t,\ P_x = -\partial p/\partial x,\ B,\ E_x,\ E_y,\ E_z,\ j,\ \partial p/\partial z,\ \rho_e$$

as characteristic dimensions for

$$a,\ U_0,\ a/U_0,\ \rho U_0^2/a,\ B_0,\ E_0,\ U_0 B_0,\ \sigma_0 E_0,\ \sigma_0 E_0 B_0,\ \varepsilon_0 B_0 U_0/a.$$

Then, writing down the components of the viscous stress tensor with the help of the formulas of [1], we obtain for the components of the equation of motion

$$\frac{\partial u_x}{\partial t} - \frac{1}{R}\frac{\partial}{\partial z}\left[\frac{\eta_1 B_y^2 + \eta_2 B_x^2}{B^2}\frac{\partial u_x}{\partial z} - \frac{B_x B_y(\eta_1 - \eta_2)}{B^2}\frac{\partial u_y}{\partial z}\right] = P_x,$$

$$\frac{\partial u_y}{\partial t} + \frac{1}{R}\frac{\partial}{\partial z}\left[\frac{B_x B_y(\eta_1 - \eta_2)}{B^2}\frac{\partial u_x}{\partial z} - \frac{\eta_1 B_x^2 + \eta_2 B_y^2}{B^2}\frac{\partial u_y}{\partial z}\right] = 0,$$

$$\frac{\partial p}{\partial z} = \frac{1}{GM^2}\frac{\partial}{\partial z}\left[\frac{\eta_3}{B}\left(B_y\frac{\partial u_x}{\partial z} - B_x\frac{\partial u_y}{\partial z}\right)\right] + j_x B_y - j_y B_x$$

$$\left(B^2 = B_x^2 + B_y^2,\ R = \frac{U_0 a\rho}{\eta^{(0)}},\ M^2 = B_0^2 a^2\frac{\sigma_0}{\eta^{(0)}},\right.$$

$$\left.G = \frac{E_0}{U_0 B_0},\ \eta_k = \frac{\eta^{(K)}}{\eta^{(0)}};\ k = 1, 2, 3\right) \qquad (1.2)$$

The dimensionless pressure differential P_x may depend on time only in the general case; in addition, it is assumed for simplicity that $\delta p/\delta y \equiv 0$. The formulas for η_k – the reduced coefficients for the viscosity of a partially ionized gas in a strong magnetic field [1], relative to the viscosity coefficient $\eta^{(0)}$ = const of gas dynamics (B = 0) – have the form

*In [5] it is assumed that j_z = const ≠ 0, which contradicts the supposition that the magnetic field depends only on the transverse coordinate.

$$\eta_1(B) = \eta_2(2B) = \frac{1 + {}^{16}/_9\,\omega_i^2\tau_i^2 B^2 \varepsilon_a}{1 + {}^{16}/_9\,\omega_i^2\tau_i^2 B^2}, \qquad \eta_3 = \frac{{}^4/_3\,\omega_i\tau_i B(1-\varepsilon_a)}{1 + {}^{16}/_9\,\omega_i^2\tau_i^2 B^2},$$
$$\left(\omega_i\tau_i = \frac{ZeB_0}{m_i}\tau_i, \qquad \varepsilon_a = \eta_a/\eta^{(0)}\right), \tag{1.3}$$

where the dimensionless parameter $\omega_i\tau_i$ characterizes the anisotropy of the viscosity coefficients (m_i is the mass, Ze is the ionic charge; τ_i is associated with the time between all possible ion collisions*). The dependence of the reduced viscosity coefficient for "isolated" neutrals ε_a on the degree of ionization for Z = 1 may be represented by the approximate formula

$$\varepsilon_a \sim \frac{(1-s)(s+\beta)^2}{[s(1-s)+\beta(1+s^2)]\,[s(1+\beta)+\beta]}$$
$$(\beta \sim 10^{-2} - 10^{-5}) \tag{1.4}$$

in accordance with the estimates of [6].

We see from (1.3) and (1.4) that for both s = 0 (ordinary hydrodynamics) and for $\omega_i\tau_i \ll 1$ (absence of Larmor gyration of ions) we have $\eta_1 = \eta_2 = 1$, $\eta_3 = 0$, and the first two equations of system (1.2) assume the form

$$\frac{\partial u_x}{\partial t} - \frac{1}{R}\frac{\partial^2 u_x}{\partial z^2} = P_x, \qquad \frac{\partial u_y}{\partial t} - \frac{1}{R}\frac{\partial^2 u_y}{\partial z^2} = 0. \tag{1.5}$$

Because of the homogeneity of the initial and boundary conditions for u_y both for Couette flow $P_x = 0$ and in flow with a given pressure gradient we obtain $u_y \equiv 0$. Thus, the presence of a coplanar magnetic field with $s \neq 0$ leads only to the appearance of a gradient dp/dz, balancing the pondermotive force. Taking account of the anisotropy of the viscosity coefficients introduces an additional contribution to the gradient mentioned as a consequence of the appearance of a transverse viscous force, and also gives rise to a motion of the medium in a direction orthogonal to the applied external force.

The current density components j_x and j_y are determined from the components of the generalized Ohm's law on the x and y axes

$$j_x = \frac{1}{1 + 2(1-s)^2\omega_i\tau_{ia}\omega_e\tau_0 B^2} \times$$
$$\times \Big\{E_x - \frac{Zs\omega_e\tau_0}{(1+Zs)GN}P_x + 2(1-s)^2\omega_i\tau_{ia}\omega_e\tau_0\Big[B_x^2E_x +$$
$$+ B_xB_yE_y + \frac{Zs}{1+Zs}\Big(B_y\frac{\partial p}{\partial z} - \frac{\omega_e\tau_0}{GN}B_x^2P_x\Big) -$$
$$- \frac{B_y}{(1-s)GM^2}\frac{\partial}{\partial z}\Big(\frac{s\eta_3 - \eta_3^i}{B}\Big(B_x\frac{\partial u_y}{\partial z} - B_y\frac{\partial u_x}{\partial z}\Big)\Big)\Big]\Big\},$$
$$j_y = \frac{1}{1 + 2(1-s)^2\omega_i\tau_{ia}\omega_e\tau_0 B^2} \times$$
$$\times \Big\{E_y + 2(1-s)^2\omega_i\tau_{ia}\omega_e\tau_0\Big[B_xB_yE_x + B_y^2E_y -$$
$$- \frac{Zs}{1+Zs}\Big(B_x\frac{\partial p}{\partial z} + \frac{\omega_e\tau_0}{GN}B_xB_yP_x\Big) +$$
$$+ \frac{B_x}{(1-s)GM^2}\frac{\partial}{\partial z}\Big(\frac{s\eta_3 - \eta_3^i}{B}\Big(B_x\frac{\partial u_y}{\partial z} - B_y\frac{\partial u_x}{\partial z}\Big)\Big)\Big]\Big\}$$
$$\left(\omega_e\tau_0 = \frac{eB_0}{m_e}\tau_0, \; N = \frac{B_0^2\sigma_0 a}{\rho U_0}\right). \tag{1.6}$$

*Here τ_i corresponds to the quantity $\tau_i\theta$ of [1].

The third component of Ohm's law gives an expression for the transverse induced electric field

$$E_z = u_yB_x - u_xB_y + \omega_e\tau_0G\Big(j_xB_y - j_yB_x - \frac{Zs}{1+Zs}\frac{\partial p}{\partial z}\Big) -$$
$$- 2(1-s)^2\omega_i\tau_{ia}\omega_e\tau_0\Big\{\frac{Zs}{(1+Zs)N}B_yP_x - \frac{B_y}{(1-s)M^2}\frac{\partial}{\partial z} \times$$
$$\times \Big[\frac{(s\eta_1 - \eta_1^i)B_y^2 + (s\eta_2 - \eta_2^i)B_x^2}{B^2}\frac{\partial u_x}{\partial z} - \frac{B_xB_y}{B^2} \times$$
$$\times (s\eta_1 - \eta_1^i - s\eta_2 + \eta_2^i)\frac{\partial u_y}{\partial z}\Big] +$$
$$+ \frac{B_x}{(1-s)M^2}\frac{\partial}{\partial z}\Big[\frac{(s\eta_1 - \eta_1^i)B_x^2 + (s\eta_2 - \eta_2^i)B_y^2}{B^2}\frac{\partial u_y}{\partial z} -$$
$$- \frac{B_xB_y}{B^2}(s\eta_1 - \eta_1^i - s\eta_2 + \eta_2^i)\frac{\partial u_x}{\partial z}\Big]\Big\}$$
$$(\eta_k^i = \eta_i^{(k)}/\eta^{(0)}, \; k = 0, 1, 2, 3). \tag{1.7}$$

The formulas for the reduced viscosity coefficients for the ionic component of a partially ionized gas η_k^i have the form [1,6]

$$\eta_1^i(B) = \eta_2^i(2B) = \frac{\eta_0^i}{1 + {}^{16}/_9\,\omega_i^2\tau_i^2B^2}, \quad \eta_3^i = \frac{{}^4/_3\,\omega_i\tau_i B\eta_0^i}{1 + {}^{16}/_9\,\omega_i^2\tau_i^2B^2},$$
$$\eta_0^i|_{Z=1} \sim \frac{s\beta(1+s)}{s(1-s) + \beta(1+s^2)}. \tag{1.8}$$

The system (1.2), (1.6), (1.7) is closed by Maxwell's equations

$$\frac{\partial B_x}{\partial t} = G\frac{\partial E_y}{\partial z}, \quad \frac{\partial B_y}{\partial t} = -G\frac{\partial E_x}{\partial z}, \quad R_mGj_x = -\frac{\partial B_y}{\partial z},$$
$$R_mGj_y = \frac{\partial B_x}{\partial z}, \quad \rho_e = \frac{\partial E_z}{\partial z} \quad (R_m = \mu_0\sigma_0 aU_0). \tag{1.9}$$

We note, in conclusion to the above paragraph, that if we have $E_x = E_y \equiv 0$, in (1.6), then for $s \neq 1$ the current j_y (and j_x for $P_x = 0$) appears as a result of the "slipping" of ions relative to neutrals, which occurs for $\omega_i\tau_{ia} \geq 1$. In this case, we take into account the contribution to the current determined by terms with velocity derivatives. However, it is not difficult to show, with the aid of estimates (1.4) and (1.8), that the coefficients η_3 and $s\eta_3 - \eta_3^i$, which appear as factors in front of the viscosity terms in (1.6), are small. Actually, $\varepsilon a \sim 1$ right up to high degrees of ionization ($s \sim 0.9$), which is explained by the predominant influence of the neutral component of the mixture on the viscosity [6]. We see from (1.3) that in this interval of variation of degree of ionization the value of η_3 is small. With further increase of s the quantity ε_a falls sharply to zero for s = 1, but for $s \sim 1$ all the parts of Ohm's law associated with the ion "slipping" effect are small. An estimate of the coefficient $s\eta_3 - \eta_3^i$ for Z = 1, $\beta = 10^{-3}$ shows that its maximum value for $\omega_i\tau_iB = 0.75$ does not exceed $\sim 5 \cdot 10^{-4}$ Thus, in the problem under consideration, the formulas (1.6) may be considered as approximately algebraic, if we neglect the influence of the velocity differential on the current in Ohm's law.*

*It does not follow from what has been said that it is always possible to neglect the viscosity term

$$\frac{2(1-s)\,\omega_i\tau_{ia}\omega_e\tau_0}{B^2}(s\,\mathrm{div}\,\pi - \mathrm{div}\,\pi_i) \times B$$

in Ohm's law.

For example, the coefficient $s\eta_0 - \eta_0^i$ for Z = 1, $\beta = 10^{-3}$ turns out to have a magnitude $\sim s$ right up to $s \sim 0.9$. Similarly, the viscous term is not small if the differences $s\eta_1 - \eta_1^i$ or $s\eta_2 - \eta_2^i$ enter into Ohm's law.

2. Steady flow. In the stationary case ($\partial/\partial t = 0$) the first two equations of (1.2) can be integrated

$$u_x = \int_0^z \Big[(C_1 - P_x Rz)\Big(\frac{B_x^2}{\eta_2} + \frac{B_y^2}{\eta_1}\Big) + C_2\Big(\frac{1}{\eta_2} - \frac{1}{\eta_1}\Big) B_x B_y\Big]\frac{dz}{B^2} + C_3,$$

$$u_y = \int_0^z \Big[(C_1 - P_x Rz)\Big(\frac{1}{\eta_2} - \frac{1}{\eta_1}\Big) B_x B_y + C_2\Big(\frac{B_x^2}{\eta_1} + \frac{B_y^2}{\eta_2}\Big)\Big]\frac{dz}{B^2} + C_4, \qquad (2.1)$$

where the arbitrary constants C_1, C_2, C_3, and C_4 must be determined for $P_x \neq 0$ from the boundary conditions

$$u_x(\pm 1) = u_y(\pm 1) = 0 \qquad (2.2)$$

and for Couette flow ($P_x = 0$) from the conditions

$$u_x(1) = 1, \quad u_x(-1) = u_y(\pm 1) = 0. \qquad (2.3)$$

It is not difficult to see that here the electrodynamic problem becomes separate from the hydrodynamic, and one may regard the dependences $B_x(z)$ and $B_y(z)$ as given. Actually, from Maxwell's first equation (1.9) and the boundary condition of continuity of the tangential components of the electric field strength vector we have

$$E_x = E_{0x} = \text{const}, \qquad E_y = E_{0y} = \text{const}, \qquad (2.4)$$

where E_{0x} and E_{0y} are the dimensionless components of the given external electric field. In addition, only velocity derivatives enter into the components of Ohm's law (1.6), and expressions for these are known from (2.1) through B_x and B_y. Thus, replacing j_x and j_y in (1.6) by derivatives of B_x and B_y from Maxwell's second equation (1.9) and the transverse pressure differential dp/dz by the third equation of system (1.2), and eliminating the velocity derivatives, we obtain for $B_x(z)$ and $B_y(z)$ a system of two nonlinear equations of the first order. Their solution for the appropriate boundary conditions gives the full solution of the problem under consideration. The system assumes a very simple form for the case of a fully ionized medium ($s = 1$)

$$\frac{dB_x}{dz} = R_m G E_{0y} = k_1 = \text{const}$$
$$\frac{dB_y}{dz} = -R_m\Big(GE_{0x} - \frac{\omega_e \tau_0}{2N} P_x\Big) = k_2 = \text{const}. \qquad (Z = 1) \quad (2.5)$$

From which

$$B_x = k_1 z + b_1, \qquad B_y = k_2 z + b_2. \qquad (2.6)$$

Here the magnetic field components are linear over the channel cross section, and, moreover, the coefficients k_1 and k_2 are determined by the constant currents flowing in the channel. Integrating Maxwell's second equation, we obtain

$$B_x(+1) - B_x(-1) = R_m G I_y = 2k_1$$
$$B_y(+1) - B_y(-1) = -R_m G I_x = 2k_2 \qquad (2.7)$$

where I_x and I_y are the components of the total current on the coodinate axes. From (2.7) it follows that in order to determine the arbitrary constants b_1 and b_2 it is sufficient that the magnetic field on one of the walls be given. For example, considering the upper plate as an insulator and giving it an external uniform field described by the formulas

$$B_{0x} = \cos\varphi, \qquad B_{0y} = \sin\varphi, \qquad (2.8)$$

where φ is the angle between the direction of B_0 and the x axis, we find

$$b_1 = \cos\varphi - k_1, \qquad b_2 = \sin\varphi - k_2. \qquad (2.9)$$

We set η_1 and η_2 from formulas (1.3) for $s = 1$ in (2.1), and also $B_x(z)$ and $B_y(z)$ from formulas (2.6); then integrating and determining the arbitrary constants C_1, C_2, C_3, and C_4 with the help of boundary conditions (2.2) and (2.3), for the velocity components of a fully ionized medium we obtain

1) for Couette flow

$$u_x(z) = {}^1/_2\,[A_1(1 + z) + A_2(1 - z^2) + A_3(1 + z^3)]$$
$$u_y(z) = {}^1/_2\,[1 - z^2]\,(A_4 + A_5 z) \qquad (2.10)$$

2) for flow due to a constant pressure differential

$$u_x(z) = {}^1/_2 P_x R\,(1 - z^2)\,(D_1 + D_2 z + D_3 z^2)$$
$$u_y(z) = {}^1/_2 P_x R\,(1 - z^2)\,(D_4 + D_5 z + D_6 z^2) \qquad (2.11)$$

The dependence of constants A_k and D_k on the anisotropy parameter for voscosity $\omega_i \tau_i$ is given by the formulas

$$A_1 = \Delta^{-1}\{1 + \omega_i^2\tau_i^2[\alpha_3 + ({}^1/_3\beta_1 + \beta_3)] + \omega_i^4\tau_i^4[\alpha_3({}^1/_3\beta_1 + \beta_3) - \gamma_3({}^1/_3\gamma_1 + \gamma_3)]\}$$
$$A_2 = -{}^1/_2\,\omega_i^2\tau_i^2\Delta^{-1}\{\alpha_2 + \omega_i^2\tau_i^2[\alpha_2({}^1/_3\beta_1 + \beta_3) - \gamma_2({}^1/_3\gamma_1 + \gamma_3)]\}$$
$$A_3 = {}^1/_3\,\omega_i^2\tau_i^2\Delta^{-1}\{\alpha_1 + \omega_i^2\tau_i^2[\alpha_1({}^1/_3\beta_1 + \beta_3) - \gamma_1({}^1/_3\gamma_1 + \gamma_3)]\}$$
$$A_4 = {}^1/_2\,\omega_i^2\tau_i^2\Delta^{-1}\{\gamma_2 + \omega_i^2\tau_i^2[\gamma_2({}^1/_3\beta_1 + \beta_3) - \beta_2({}^1/_3\gamma_1 + \gamma_3)]\}$$
$$A_5 = {}^1/_3\,\omega_i^2\tau_i^2\Delta^{-1}\{\gamma_1 - \omega_i^2\tau_i^2(\beta_1\gamma_3 - \beta_3\gamma_1)\}$$
$$D_1 = 1 + \omega_i^2\tau_i^2({}^1/_2\alpha_1 + \alpha_3) - {}^1/_3\omega_i^4\tau_i^4\Delta^{-1}\{\alpha_2^2 + \gamma_2^2 + \omega_i^2\tau_i^2 \cdot [\alpha_2^2({}^1/_3\beta_1 + \beta_3) - 2\alpha_2\gamma_2({}^1/_3\gamma_1 + \gamma_3) + \gamma_2^2({}^1/_3\alpha_1 + \alpha_3)]\}$$
$$D_2 = {}^2/_3\omega_i^2\tau_i^2\Delta^{-1}\{\alpha_2 + \omega_i^2\tau_i^2[\alpha_2\alpha_3 + \alpha_2({}^1/_3\beta_1 + \beta_3) - {}^1/_3\gamma_1\gamma_2] + \omega_i^4\tau_i^4[\alpha_2\alpha_3({}^1/_3\beta_1 + \beta_3) + {}^1/_3\gamma_2(\alpha_1\gamma_3 - \alpha_3\gamma_1) - \alpha_2\gamma_3({}^1/_3\gamma_1 + \gamma_3)]\}$$
$$D_4 = -\omega_i^2\tau_i^2({}^1/_2\gamma_1 + \gamma_3) + {}^1/_3\omega_i^4\tau_i^4\Delta^{-1}\{\gamma_2(\alpha_2 + \beta_2) + \omega_i^2\tau_i^2 \cdot [-(\alpha_2\beta_2 + \gamma_2^2)({}^1/_3\gamma_1 + \gamma_3) + \beta_2\gamma_2({}^1/_3\alpha_1 + \alpha_3) + \alpha_2\gamma_2({}^1/_3\beta_1 + \beta_3)]\}$$
$$D_5 = -{}^2/_3\,\omega_i^2\tau_i^2\Delta^{-1}\{\gamma_2 + \omega_i^2\tau_i^2[\beta_3\gamma_2 + \gamma_2({}^1/_3\alpha_1 + \alpha_3) - {}^1/_3\alpha_2\gamma_1] + \omega_i^4\tau_i^4[\beta_3\gamma_2({}^1/_3\alpha_1 + \alpha_3) + {}^1/_3\alpha_2(\beta_1\gamma_3 - \beta_3\gamma_1) - \gamma_2\gamma_3({}^1/_3\gamma_1 + \gamma_3)]\}$$
$$D_3 = {}^1/_2\,\omega_i^2\tau_i^2\alpha_1, \qquad D_6 = -{}^1/_2\,\omega_i^2\tau_i^2\gamma_1$$
$$\Delta = 1 + \omega_i^2\tau_i^2[({}^1/_3\alpha_1 + \alpha_3) + ({}^1/_3\beta_1 + \beta_3)] + \omega_i^4\tau_i^4[({}^1/_3\alpha_1 + \alpha_3)({}^1/_3\beta_1 + \beta_3) - ({}^1/_3\gamma_1 + \gamma_3)^2]. \qquad (2.12)$$

Finally, the constants α_k, β_k and γ_k have the form

$$\alpha_1 = {}^4/_9(k_1^2 + 4k_2^2), \qquad \alpha_2 = {}^8/_9(k_1 b_1 + 4k_2 b_2), \qquad \alpha_3 = {}^4/_9(b_1^2 + 4b_2^2)$$
$$\beta_1 = {}^4/_9(4k_1^2 + k_2^2), \qquad \beta_2 = {}^8/_9(4k_1 b_1 + k_2 b_2), \qquad \beta_3 = {}^4/_9(4b_1^2 + b_2^2)$$
$$\gamma_1 = {}^4/_3\,k_1 k_2, \qquad \gamma_2 = {}^4/_3(k_1 b_2 + k_2 b_1), \qquad \gamma_3 = {}^4/_3\,b_1 b_2 \qquad (2.13)$$

Assuming $\omega_i \tau_i \ll 1$ (absence of viscosity anisotropy), we obtain from (2.10) and (2.11), respectively,

$$u_x = {}^1/_2(1+z),\ u_y \equiv 0; \qquad u_x = {}^1/_2 P_x R(1-z^2),\ u_y \equiv 0,$$

i.e., the ordinary hydrodynamic modes.

It should also be noted that if, in the case under consideration, one of the magnetic induction vector components is equal to zero, then the effect associated with transverse motion of the medium does not occur. Actually, if k_1, b_1 or k_2, b_2 vanish simultaneously, then $\gamma_1=\gamma_2=\gamma_3=0$, whence $A_4=A_5=D_4=D_5=$ $=D_6=0$ and $u_y\equiv 0$. For example, $k_1=b_1=0$, when $E_{0y}=0$ and $\varphi=\pi/2$. However, in this case, taking into account the viscosity anisotropy radically distorts the hydrodynamic velocity profiles.

3. Nonsteady flow. When formulated exactly, the nonstationary problem is complicated to investigate. It does, however, admit of a simple exact solution for the case of a fully ionized medium. In fact, eliminating the components of current density and electric field from Ohm's law (1.6) for s =1 with the help of Maxwell's equations (1.9), we obtain the following induction equations:

$$\frac{\partial B_x}{\partial t}=\frac{1}{R_m}\frac{\partial^2 B_x}{\partial z^2},\qquad \frac{\partial B_y}{\partial t}=\frac{1}{R_m}\frac{\partial^2 B_y}{\partial z^2}. \tag{3.1}$$

If, for simplicity, we consider that the total currents through the channel cross section are equal to zero, then on the basis of (2.7) we may put the boundary and initial conditions in the form

$$\begin{aligned} B_x(z,0) &= B_x(\pm 1,t)=B_{0x}=\cos\varphi,\\ B_y(z,0) &= B_y(\pm 1,t)=B_{0y}=\sin\varphi, \end{aligned} \tag{3.2}$$

from which we have at once that the magnetic field is identically equal to the homogeneous external field. From Eqs. (1.2), assuming s =1, B = 1, we then obtain

$$\begin{aligned} &R\frac{\partial u_x}{\partial t}-(\eta_1\sin^2\varphi+\eta_2\cos^2\varphi)\frac{\partial^2 u_x}{\partial z^2}+\\ &+(\eta_1-\eta_2)\sin\varphi\cos\varphi\frac{\partial^2 u_y}{\partial z^2}=P_xR\\ &R\frac{\partial u_y}{\partial t}+(\eta_1-\eta_2)\sin\varphi\cos\varphi\frac{\partial^2 u_x}{\partial z^2}-\\ &-(\eta_1\cos^2\varphi+\eta_2\sin^2\varphi)\frac{\partial^2 u_y}{\partial z^2}=0, \end{aligned} \tag{3.3}$$

where

$$\eta_1=\frac{1}{1+{}^{16}/_9\,\omega_i^2\tau_i^2},\qquad \eta_2=\frac{1}{1+{}^4/_9\,\omega_i^2\tau_i^2}. \tag{3.4}$$

System (3.3) may be solved for arbitrary φ using a Laplace transform. However, for $\varphi=0$ (magnetic field parallel to the direction of the external force causing the motion), $\varphi=\pi/2$ (magnetic field perpendicular to this force), and $\varphi=\pi/4$ the system has simple solutions. Indeed, for $\varphi=0$ or $\varphi=\pi/2$ the system (3.3) assumes the form (1.5), but with an effective Reynolds number R*

$$R^*=\begin{cases} R_1^*=R/\eta_1 & \text{for } \varphi={}^1/_2\pi\\ R_2^*=R/\eta_2 & \text{for } \varphi=0. \end{cases} \tag{3.5}$$

Thus, the solution of the Couette problem for boundary conditions (2.3) and a homogeneous initial condition has the form

$$u_x(z,t)=\frac{1+z}{2}-\sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda_n}\cos\lambda_n z\exp\frac{-\lambda_n^2 t}{R^*},\quad u_y\equiv 0$$
$$(\lambda_n={}^1/_2(2n+1)\pi). \tag{3.6}$$

It is clear from (3.6) that a homogeneous magnetic field merely delays the formation of the stationary mode because of the lessening of the viscosity of the medium when the effect of anisotropy of viscosity is taken into account, and it also follows from (3.4) and (3.5) that a magnetized medium $\omega_i\tau_i \gg 1$ will have very low viscosity.

Similarly, for flow resulting from a constant pressure differential P_x we have, for homogeneous initial and boundary conditions,

$$u_x(z,t)=\frac{P_xR^*}{2}(1-z^2)-2P_xR^*\sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda_n^3}\cos\lambda_n z\exp\frac{-\lambda_n^2 t}{R^*}$$
$$U_y\equiv 0 \tag{3.7}$$

Here the lessening of the viscosity of the medium in a strong magnetic field also affects the steady-state mode and is accompanied by a general increase of velocity over the channel cross section for values of $\omega_i\tau_i$ which are not small.

Finally, for $\varphi=\pi/4$ system (3.3) can also be transformed to the form (1.5) by the substitution

$$u_x={}^1/_2(v_1+v_2),\qquad u_y={}^1/_2(v_1-v_2) \tag{3.8}$$

where (3.3) is rewritten as

$$\frac{\partial v_{1,2}}{\partial t}-\frac{1}{R_{2,1}^*}\frac{\partial^2 v_{1,2}}{\partial z^2}=P_x. \tag{3.9}$$

Thus, in the Couette problem we have

$$\begin{aligned} u_x(z,t)&=\frac{1+z}{2}-\sum_{n=0}^{\infty}\frac{(-1)^n}{2\lambda_n}\cos\lambda_n z\left(\exp\frac{-\lambda_n^2 t}{R_1^*}+\exp\frac{-\lambda_n^2 t}{R_2^*}\right)\\ u_y(z,t)&=\sum_{n=0}^{\infty}\frac{(-1)^n}{2\lambda_n}\cos\lambda_n z\left(\exp\frac{-\lambda_n^2 t}{R_1^*}-\exp\frac{-\lambda_n^2 t}{R_2^*}\right) \end{aligned} \tag{3.10}$$

in the problem with a constant pressure differential

$$\begin{aligned} u_x(z,t)&=\frac{P_x(R_1^*+R_2^*)}{4}(1-z^2)-P_x\sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda_n^3}\cos\lambda_n z\times\\ &\times\left(R_1^*\exp\frac{-\lambda_n^2 t}{R_1^*}+R_2^*\exp\frac{-\lambda_n^2 t}{R_2^*}\right)\\ u_y(z,t)&=-\frac{P_x(R_1^*-R_2^*)}{4}(1-z^2)+P_x\sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda_n^3}\cos\lambda_n z\times\\ &\times\left(R_1^*\exp\frac{-\lambda_n^2 t}{R_1^*}-R_2^*\exp\frac{-\lambda_n^2 t}{R_2^*}\right). \end{aligned} \tag{3.11}$$

In conclusion, we note that if conditions exist such that one may neglect the induced magnetic field in comparison with the given uniform external field, then the solutions given for system (3.3) are valid in the zero-th approximation even for s ≠ 1, only (1.3) must be used instead of (3.4). We shall find these conditions, and to do this we turn to the equations of the problem in dimensional notation (1.1). Assuming $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}^*(z, t)$ (where B_0 = const is the external field, and B^* is the induction field), we see from Maxwell's second equation that $B^* \sim \sim \mu_0 a j$. To determine the characteristic quantity j we estimate the relative contributions to the conduction current made by the electric and nonelectric forces in Ohm's law.

$$\frac{(1+Zs)\,B\sigma_0 E}{Zs\omega_e\tau_0\nabla p} \sim (1+Zs)\,\frac{eE_0 a}{m_i U_0^2} = (1+Zs)\,\frac{\omega_i G}{Z\Omega}$$

$$\left(\sigma_0 = \frac{Zspe^2\tau_0}{m_e m_i}, \; \nabla p \sim \frac{\rho U_0^2}{a}, \; E \sim E_0, \right.$$

$$\left. \Omega = \frac{U_0}{a}, \; G = \frac{E_0}{U_0 B_0}\right). \qquad (3.12)$$

Since we are considering a slow process (ρ = = const), it is obvious that the contribution to the conduction current from the pressure gradient will be of the same magnitude as the contribution from the electric force only in very weak electric fields.

For example, for $\Omega = 1\ \mathrm{sec}^{-1}$ and $\omega_i / Z \approx 10^8\ \mathrm{sec}^{-1}$ ($B_0 = 10^4$ G) we should have $G \sim 10^{-8}$ (for U_0 = = 1 m/sec we have $E_0 \sim 10^8$ V/m).

For values of E, which are not small the term $c\nabla p$ in Ohm's law may be neglected regardless of s (in the case under consideration the $u \times B$ field does not enter into the equations for the current). In addition to this, on comparing the j term in Ohm's law with the slip current $B \times (j \times B)$, we see that the latter is $2(1-s)^2\omega_i\tau_{ia}\omega_e\tau_0$ times greater than the former and for $s \neq 1$ and $\omega_i\tau_{ia} \geq O(1)$ $\omega_e\tau_0/\omega_i\tau_{ia} \sim \sim \sqrt{m_i/m_e} \gg 1$ is the determining factor (in a coplanar magnetic field the Hall current does not have a component in the plane of flow). Thus for an electric field which is not very weak $\omega_i G/Z\Omega \gg 1$ $s \neq 1$ and $\omega_i\tau_{ia} \geq \geq O(1)$; from the estimates which have been made we find

$$j \sim \frac{\sigma_0 E_0}{2(1-s)^2\omega_e\tau_0\omega_i\tau_{ia}}, \quad B^* \sim \frac{R_m G}{2(1-s)^2\omega_e\tau_0\omega_i\tau_{ia}} B_0. \qquad (3.13)$$

Hence $B^* \ll B_0$, if

$$\frac{R_m G}{2(1-s)^2\omega_e\tau_0\omega_i\tau_{ia}} \ll 1. \qquad (3.14)$$

For s = 1 the slip current vanishes, and we have $R_m G \ll 1$ instead of (3.14).

If there is no external electric field, and the induced field E^* is so small that the current is determined by the pressure gradient, then

$$2(1-s)^2\omega_i\tau_{ia} j \sim \frac{Zs}{1+Zs}\frac{\rho U_0^2}{aB_0},$$

$$B^* \sim \frac{Zs}{2(1+Zs)(1-s)^2\omega_i\tau_{ia}\Pi} B_0$$

$$(\Pi = B_0^2/\mu_0\rho U_0^2) \qquad (3.15)$$

We note that for the flow geometry under consideration the term $\nabla p \times B$ is determined by the current and is eliminated from Ohm's law with the aid of the equation of motion.

From (3.15) we have $B^* \ll B_0$ for $s \neq 1$ if

$$\frac{Zs}{2(1+Zs)(1-s)^2\omega_i\tau_{ia}\Pi} \ll 1. \qquad (3.16)$$

Moreover, it follows from (3.12) that one must also require that

$$E^* \ll \frac{m_i U_0^2}{(1+Zs)ea}. \qquad (3.17)$$

Since here in the nonstationary case E_x^* and $E_y^* \sim U_0 B^*$, from (3.17) and (3.15) we obtain the condition supplementary to (3.16), ensuring the smallness of E^*

$$\frac{s\tau_{ia}^{-1}}{2(1-s)^2\Pi\Omega} \ll 1. \qquad (3.18)$$

REFERENCES

1. E. G. Sakhnovskii, "Single fluid dynamic equations of a partially ionized gas in a strong magnetic field," PMM, vol. 28, no. 3, 1964.

2. E. G. Sakhnovskii, "Nonstationary plane-parallel flow of a partially ionized gas in a strong magnetic field " PMM, vol. 28, no. 4, 1964.

3. A. I. Gubanov and Yu. P. Lun'kin, "Couette flow in magnetoplasmadynamics," Zh. Tekh. Fiz. vol. 30, no. 9, 1960.

4. V. B. Baranov, "The influence of a magnetic field on temperature distribution in Couette flow in anisotropic magnetohydrodynamics," PMTF, no. 6, 1962.

5. E. P. Zimin, "Flow of an electrically conducting fluid in a magnetohydrodynamic channel," PMTF, no. 6, 1963.

6. M. Ya. Alievskii, V. M. Zhdanov, and V. A. Polyanskii, "The viscous stress tensor and heat flow in a two-temperature, partially ionized gas," PMTF, no. 3, 1964.

31 January 1965 Leningrad